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COMMON FIXED POINT THEOREM FOR SELF MAP SATISFYING RATIONAL TYPE CONTERACTIVE CONDITON IN CONE RECTANGULAR METRIC SPACE

A. KUMAR
Department of Mathematics
D.A.V. College, Kanpur (U.P.) India
Email: sri.abhishek66@gmail.com

ABSTRACT

In this paper we have proved a common fixed point theorem by using rational type contraction conditionwhich extends the result of Jaleli and Bessem Samet [6].

Keywords: Cone metric space, cone rectangular metric space, Cauchy sequence and ordered Banach space.

200 AMS Subject Classification: 46EXX, 46E35, 47HXX, 47H10.

Introduction

Huang and Zhang [5] have introduced by an ordered Banach space, and they established some fixed piont theorems for contractive type mappings in a normal cone metric space.

The idea of Branciari [3], Azam, Arshad and Beg [2] extended the notion of cone metric spaces by replacing the triangular inequality by a rectangular inequality. The aim of this paper is to extend the result of Jaleli and Bessem Samet [6] in such spaces.

Definition 1. If E be a real Banach space and a nonempty set X . Suppose that the mapping satisfies

$$0 < d(x,y) \text{ for all } x,y \in X$$

$$d(x,y) = 0 \text{ if and only if } x = y$$

$$d(x,y) = d(y,x) \text{ for all } x,y \in X$$

$$d(x,y) \le d(x,z) + d(y,z) \text{ for all } x,y,z \in X.$$

Then distance d is called a cone metric on X and set X with cone metric d is called cone metric space (X,d).

Definition 2. let X be a nonempty set and E be a real Banach space. Suppose the mapping $d : X \times X \to E$ satisfies

$0 \le d(x,y), \forall x,y \in X$.

$d(x,y) = 0$ if and only if $x = y$.

$d(x,y) = d(y,x)$ $\forall x,y \in X$.

$$d(x,y) \le d(x,w) + d(w,z) + d(z,y), \forall x,y \in X$$

and for all distinct point $w,z \in X - \{x,y\}$ (rectangular property).

Then d is called a cone rectangular metric on X and (X,d) is called cone rectangular metric space.

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Lemma 2.1.A sequence $\{x_n\}$ in cone rectangular metric space X is said to be convergent if for every $c \in E$ with $0 << c$ there is $n_0 \in N$ such that

$$d(x_n, x) << c \text{ for all } n > n_0.$$

Lemma 2.2. A sequence $\{x_n\}$ in cone rectangular metric space X is said to be Cauchy if for every $c \in E$ with $0 << c$ there is $n_0 \in N$ such that

$$d(x_n, x_m) << c \text{ for all } n, m > n_0.$$

If every Cauchy sequence in cone rectangular metric space X is convergent then X is said to be complete cone rectangular metric space.

For $c \in E$ with $0 << c$ there is $n_0 \in N$ such that for all $n, m > n_0$ $d(x_n, x_m) << c$ then $\{x_n\}$ is called Cauchy sequence.

A cone rectangular metric space is said to be complete cone rectangular metric space if every Cauchy sequence in X is convergent.

Theorem 1. Let (X, d) be a complete cone rectangular metric space P be a normal cone with normal constant K. Suppose a mapping $f : X \to X$ satisfying contractive condition

$$d(fx, fy) \le a\left[\frac{d(x, fx)d(y, fy)}{d(x, y)} + d(x, fx) + d(y, fy) + d(x, y)\right] \forall x, y \in X$$

where $a \in [0, \frac{1}{5})$. Then

f has a unique fixed point in X

for any $x \in X$, the iterative sequence $f^n x$
converges to the fixed point.
Now for $x \in X$ we have

$$d(fx, f^2x) = d(fx, ffx)$$

$$\le \left[\frac{d(x, fx)d(fx, f^2x)}{d(x, fx)} + d(x, fx) + d(fx, f^2x) + d(x, fx)\right]$$

$$\le 2a\left[d(fx, f^2x) + d(x, fx)\right]$$

$$d(fx, f^2x) \le \frac{2a}{1 - 2a} d(x, fx)$$

$$d(f^2x, f^3x) = d(ffx, ff^2x)$$

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$$\leq a\left[\frac{d(fx,f^2x)d(f^2x,f^3x)}{d(fx,f^2x)} + d(fx,f^2x)d(f^2x,f^3x) + d(fx,f^2x)\right]$$

$$\leq a\left[d(f^2x,f^3x) + d(fx,f^2x) + d(f^2x,f^3x) + d(fx,f^2x)\right]$$

$$d(f^2x,f^3x) \leq \frac{2a}{1-2a}d(fx,f^2x)$$

$$\leq \left(\frac{2a}{1-2a}\right)^2 d(x,fx)$$

Thus in general, if n is a positive integer then

$$d(f^nx,f^{n+1}x) \leq \left(\frac{2a}{1-2a}\right)^n d(x,fx)$$

$$d(f^nx,f^{n+1}x) \leq k^n d(x,fx)$$

where $k = \dfrac{2a}{1-a} \in [0,1)$ we divide the proof into two case

First case. Let $f^mx = f^nx$ for some $m,n \in N$, $m \neq n$. Let $m > n$. Then $f^{m-n}(f^nx) = f^nx$ i.e. $f^py = y$ where $p = m-n$, $y = f^nx$. Now since $p > 1$ we have

$$d(y,fy) = d(f^py,f^{p+1}y)$$

$$d(y,fy) \leq k^p d(y,fy)$$

Since $k \in [0,1)$ we obtain $-d(y,fy) \in P$ and $d(y,fy) \in P$ which implies that $\|d(y,fy)\| = 0$ i.e. $fy = y$.

Secondcase. Assume that $f^mx \neq f^nx$ for all $m,n \in N$, $m \neq n$. Clearly, we have

$$d(f^nx,f^{n+1}x) \leq k^n d(x,fx)$$

$$\leq \frac{k^n}{1-k}d(x,fx)$$

and

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$$d(f^n x, f^{n+2}x) \le a\left\{(f^{n-1}x, f^n x) + d(f^{n+1}x, f^{n+2}x)\right\}$$

$$\le a\left(k^{n-1}d(x,fx) + k^{n+1}d(x,fx)\right)$$

$$\le k^n d(x,fx) + k^{n+1}d(x,fx)$$

$$\le \frac{k^n}{1-k}d(x,fx)$$

Now if $m > 2$ is odd then writing $m = 2l+1,\ l \ge 1$ and using the fact that $f^p x \ne f^r x$ for $p, r \in N,\ p \ne r$ we can easily show that

$$d(f^n x, f^{n+m}x) \le d(f^{n-1}x, f^n x) + d(f^{n+1}x, f^{n+2}x) + d(f^{n+2l-1}x, f^{n+2l}x)$$

$$\le \frac{k^n}{1-k}d(x,fx)$$

Again if $m \ge 2$ is even then writing $m = 2l \ge 2$ and using the same arguments as before we can get,

$$d(f^n x, f^{n+m}x) \le d(f^{n+1}x, f^{n+2}x) + d(f^{n+2}x, f^{n+3}x) + \cdots + d(f^{n+2l-1}x, f^{n+2l}x)$$

$$\le k^n d(x,fx) + k^{n+1}d(x,fx) + \cdots + k^{n+2l}d(x,fx)$$

$$\le \frac{k^n}{1-k}d(x,fx)$$

Thus combing all the cases we have

$$d(f^n x, f^{n+m}) \le \frac{k^n}{1-k}d(x,fx),\ \forall m,n \in N$$

Hence we get

$$\| d(f^n x, f^{n+m}x)\| \le \frac{k^n}{1-k}\| d(x,fx)\|,\ \forall m,n \in N$$

Since $K\dfrac{k^n}{1-k}\| d(x,fx)\| \to 0$ as $n \to \infty$

$\{f^n x\}$ is a Cauchy sequence.By the completeness of X there is $x \in X$ such that

$f^n x \to x$ as $n \to +\infty$

We shall now show that $fx^* = x^*$ without any loss of generality, we can assume that $fx^* \ne x^*, fx^*$ for any $r \in N$ we have

$$d(x^*, fx^*) \le d(x^*, f^{n+1}x) + d(f^{n+1}x, fx^*)$$

$$\le d(x^*, f^n x) + d(f^n x, f^{n+1}x) + a\left\{d(f^n x, f^{n+1}x) + d(x^*, fx^*)\right\}$$

$$d(x^*, fx^*) \le \frac{1}{1-a}\left\{d(x^*, f^n x) + (1+a) \times d(f^n x, f^{n-1}x)\right\}$$

Hence

$$\| d(x^*, fx^* \| \le \frac{K}{1-a}\left\{\| d(x^*, f^n x) \| + (1+a) \| d(f^n x, f^{n+1}x) \|\right\} \to 0$$

as $n \to \infty$

So we obtain $d(fx^*, x^*) = 0$ i.e. $x^* = fx^*$

Now if y^* is another fixed point of f, then

$$d(x^*, y^*) = d(fx^*, fy^*) \le a\left\{d(x^*, fx^*) + d(y^*, fy^*)\right\} = 0$$

which implies that $\| d(x^*, y^*) \| = 0$ i.e. $x^* = y^*$.

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